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## Limited quantum decay

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**Abstract.** An isolated level coupled to a continuum of levels need not decay fully into them. Ordinarily one expects exponential decay, modulo finite-size effects, transients and long-time power-law tails. In the phenomenon we describe the amplitude for remaining in the initial state at first drops, but then levels off and remains  $O(1)$  indefinitely. Because of the available continuum, this is different from certain quantum localization phenomena where there is an absence of on-shell levels. The origin of the effect we describe is the existence of thresholds and band edges. A condition relating the proximity to threshold with the strength of the coupling determines whether the limited decay occurs.

### 1. Introduction

In nature, simple quantum systems are found to decay exponentially. In theoretical physics there are famous long-time and short-time departures [1–5], but it is only recently that the short-time phenomenon has been observed [6] and the long-time power law has never been seen. On the other hand, if one models quantum decay on a computer, it may call for a good deal of fiddling with parameters before the decay looks exponential and even then there will still be a well understood Poincaré recurrence [7].

In this paper we discuss a phenomenon in which the system not only fails to have exponential behaviour, but substantially fails to decay. A single level is coupled to a continuum of decay channels, with no shortage of on-shell levels. Nevertheless, even as  $t \rightarrow \infty$ , a significant fraction of the amplitude remains in the original level. For reasons that will be apparent below, this effect is especially likely to be seen in numerical work. There is, in addition, a finite-size effect: if the decay states do not form a continuum there will be an anomalously large remnant undecayed amplitude, even for cases where the continuum limit of the model would lose that amplitude. Again, numerical modelling will be sensitive to this effect.

The absence of transitions that are ostensibly allowed both by energy and by coupling considerations makes an appearance in other physical applications. A well known phenomenon along these lines is the failure to tunnel, variously observed in a ‘quantum localization’ context [8] or even in a simple one-dimensional two-well system [9]. What we show here goes beyond the results of those authors in that we actually have a *continuum* of levels available, and *the decay is still hindered*.

The origin of this phenomenon lies in the fact that the continuum of levels to which a system can decay does not extend from  $-\infty$  to  $\infty$ . First, energy is always bounded from below. Furthermore, the continuum into which the system decays may be a band of levels, as in a solid, which is bounded both above and below. The edge can give rise to modes

of the system resistant to the decay. These modes are closely related to the *plasmon* or quasiparticle modes encountered in condensed matter physics [11, 12], although, as far as we know, time-dependent phenomena associated with these modes have not been noted. Actually, the defective decay arises not merely from the existence of these modes, but from the fact that the unstable level (the initial state, which in the absence of the decay coupling would have energy matching some of the continuum levels) has overlap with the plasmon mode that is of the order of unity, despite the fact that the overlap with most modes is a positive power of the level spacing ( $\Delta\omega$ ) and vanishes in the continuum limit. Another phenomenon related to the existence of edges is the well known extreme long-time power-law tail. However, we emphasize that the phenomenon we study here is not *slow* decay, but *failure* to decay.

Besides the plasmon modes just mentioned, there is the possibility that other physical decaying systems would satisfy the coupling constant threshold requirements that we give below and give rise to anomalously long-lived states. Conceivably, such states have already been observed, but their origin misinterpreted.

In section 2, we define our model of a level perturbed by a multitude of other levels and find the changes in the spectrum of the Hamiltonian. To study the decay, we also need the eigenfunctions, and further information on them is derived in section 3. The fourth section puts this information to use and we show that if the threshold behaviour is too sharp, a failure to decay can occur. This demonstration is the main point of this paper. We also present a derivation of the usual exponential decay based on the eigenfunction expansion. This is of interest because it does not use Laplace transform methods.

### 1.1. The numerics of decay

It would seem reasonable that if one defined a Hamiltonian in terms of a single level that could decay into a continuum of levels, without worrying too much about coupling, you

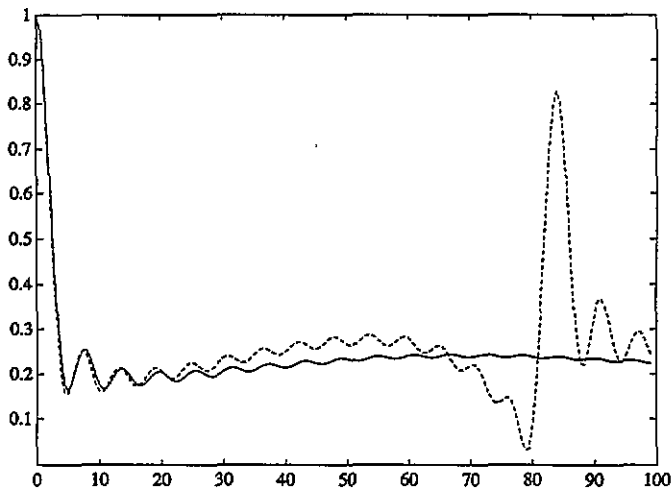


Figure 1. The norm of wavefunction remaining in the initial state as a function of time. The Hamiltonian is of the form shown in (2.1) and the initial state has  $x = 1$  (in the notation section 2). For 'C' we take  $C_k = (c/\sqrt{M})[1 - (k/(M+1))^2]^{1/4}$ ,  $k = -M, \dots, M$ , with  $c = \frac{1}{4}$  and various  $M$ . (The 'N' of section 2 is  $2M + 1$ .) ' $\Omega$ ' is diagonal with  $2M + 1$  levels, evenly spaced between 0 and 2. The offset,  $\hbar_0$ , is 0.7. The full curve corresponds to  $|\psi|^2$  for  $M = 17$  and the broken curve is  $M = 6$  (corresponding to  $36 \times 36$  and  $14 \times 14$  matrices, respectively).

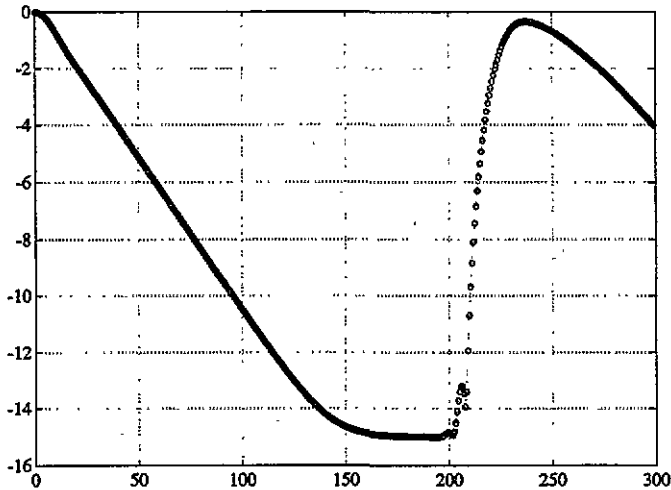


Figure 2. Logarithm of the norm of wavefunction remaining in the initial state as a function of time. The Hamiltonian has the same form as in (2.1) and is similar to that used for the previous figure. The main difference is that instead of the power  $\frac{1}{4}$ , used in  $C$  we use the power 2 (other differences:  $h_0 = 0.9$  and  $c = 0.1$ ).  $H$  is again  $36 \times 36$ .

should inevitably get exponential decay. Fermi's 'Golden Rule' suggests that only on-shell matrix elements are important and if they are not doing anything strange one should find the exponential. Here we present numerical results which show that expectation to be unfounded.

Figure 1 shows the results of a typical attempt. What is depicted is the norm of the amplitude remaining in an initial state which is in contact with a continuum. (The precise Hamiltonian is given in the figure caption.) For a while one seems to have exponential decay, but this does not empty the original state. Moreover, there remains an oscillatory time dependence. This phenomenon is *not* the Poincaré recurrence associated with the finiteness of the matrices. To show this we plot the decay curves for two different sized matrices, in one of which there is a recurrence, in one not. In this regard the two decays differ; however, the residual probability, well before the recurrence time, is essentially the same for both matrices. In figure 2 we show a successful decay. The difference between these and the reasons for exponential decay in one case and not in the other is the subject of this paper.

## 2. Interaction of a level with a continuum: the spectrum

In this section, we consider a Hamiltonian,  $H$ , that is suitable for describing the decay of a single level into a continuum. In section 2.1 we solve for the eigenvalues and eigenvectors of  $H$ , for finite-dimensional  $H$ . We then consider the continuum limit. We find that the behaviour of the largest and smallest eigenvalues can be particularly interesting and in section 2.2 we study these in a limit of high dimension. Finally in section 2.3 the continuum limit of the other eigenvalues is computed.

### 2.1. Solution of the finite-dimensional eigenvalue problem

The context is a matrix Hamiltonian describing the decay of a single level into a quasicontinuum of states. The Hilbert space of the system is  $(N + 1)$ -dimensional and

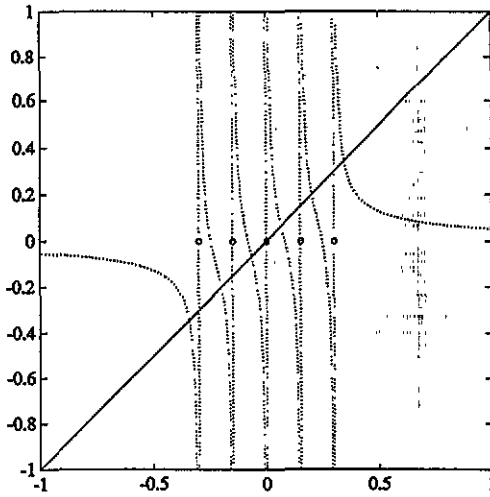


Figure 3. The curved dotted lines are the function  $\sum_{i=1}^N |c_i|^2 / (\lambda - \omega_i)$ . The full curve is the function  $\lambda$ . The straight vertical lines are for visualization.

consists of vectors of the type

$$\psi = \begin{pmatrix} x \\ Y \end{pmatrix}$$

where  $x$  is a complex number and  $Y$  is an  $N$ -dimensional complex vector with components  $(Y)_j = y_j$ . The Hamiltonian is

$$H = \begin{pmatrix} h_0 & C^\dagger \\ C & \Omega \end{pmatrix} \tag{2.1}$$

where  $h_0$  is a real number,  $\Omega$  is the diagonal matrix  $\text{diag}(\omega_1, \dots, \omega_N)$  and  $C$  is a complex  $N$ -vector with components  $(C)_j = c_j$ . The range of the spectrum of  $\Omega$ , that is, the interval  $[\omega_1, \omega_N]$ , will enter our later considerations and we designate it  $\sigma_\Omega$ . The eigenvalue problem,  $H\psi = \lambda\psi$ , is decomposed into a scalar and a vector equation

$$h_0x + C^\dagger Y = \lambda x \quad Cx + \Omega Y = \lambda Y.$$

We solve for  $Y$

$$y_i = \frac{xc_i}{\lambda - \omega_i} \tag{2.3}$$

and obtain the eigenvalue equation for  $\lambda$

$$\lambda - h_0 = \sum_{i=1}^N \frac{|c_i|^2}{\lambda - \omega_i} \tag{2.4}$$

The solutions of (2.4) can be obtained graphically in the usual way. The function

$$\sum_{i=1}^N \frac{|c_i|^2}{\lambda - \omega_i} \tag{2.5}$$

has poles at  $\lambda = \omega_i$  and tends to zero at infinity as shown in figure 3. There is a solution  $\lambda_j$  of (2.4) in each interval  $[\omega_j, \omega_{j+1}]$  for  $1 \leq j \leq N - 1$ . Moreover, we have two other solutions, one of which lies to the left of the others, one to the right. These are denoted  $\lambda_0$  and  $\lambda_\infty$ , and satisfy

$$-\infty < \lambda_0 < \omega_1 \quad \omega_N < \lambda_\infty < +\infty.$$

In figure 3 we illustrate the case where  $h_0$  is embedded in the quasicontinuum band  $(\omega_1, \dots, \omega_N)$  and when all the  $c_j$  are different from 0.

For  $j = 0, 1, 2, \dots, N - 1, \infty$ , the normalization condition on the eigenvector gives the relation

$$x^{(j)} = \frac{1}{\left(1 + \sum_{i=1}^N \frac{|c_i|^2}{(\lambda_j - \omega_i)^2}\right)^{1/2}} \tag{2.6}$$

for the  $x$  component of the eigenvector  $\psi^{(j)}$  of eigenvalue  $\lambda^{(j)}$ . The normalized eigenvector  $\psi^{(j)}$  of eigenvalue  $\lambda_j, j = 0, 1, \dots, N - 1, \infty$  has components

$$\psi^{(j)} = \begin{pmatrix} x^{(j)} \\ Y^{(j)} \end{pmatrix}$$

where the  $k$ -component of  $y^{(j)}$  follows from (2.3)

$$y_k^{(j)} = \frac{x^{(j)} c_k}{\lambda_j - \omega_k}. \tag{2.7}$$

2.2. Continuum limit of the extreme eigenvalues

In the forthcoming subsection we consider the effect on the extreme eigenvalues,  $\lambda_0$  and  $\lambda_\infty$ , of the limit  $N \rightarrow \infty$ , i.e. the continuum limit of  $H$ . We thus focus on limiting forms of (2.4). There are two principal situations:  $\sigma_\Omega$  is bounded both above and below (as  $N \rightarrow \infty$ ) or it is only bounded from below. In both cases we find that the behaviour of the external roots,  $\lambda_0$  and  $\lambda_\infty$ , in particular, whether they remain separated from the other roots, depends on the behaviour of the coupling constants and density of states near the edges.

2.2.1.  $\sigma_\Omega$  bounded above and below. We first consider the case where the interval,  $\sigma_\Omega$ , is finite. For convenience we take  $\omega_1 = 0$ . For  $\lambda$  outside  $\sigma_\Omega$  and for large  $N$ , the eigenvalue equation is well approximated by a continuum limit. We replace  $\sum_i$  in (2.4) by  $\int d\omega$ . equation (2.4) becomes

$$\lambda - h_0 = \int_0^a \frac{|\gamma(\omega)|^2 \rho(\omega)}{\lambda - \omega} d\omega$$

where  $\omega$  is a function of  $i$  defined by  $\omega = \omega_i, a = \omega_N$  and we have defined

$$\gamma(\omega) = \gamma_i = c_i \sqrt{N} \quad \rho(\omega) = \frac{1}{N(\omega_{i+1} - \omega_i)}.$$

The scaling of  $c_i, c_i \sim \text{constant}/\sqrt{N}$ , is the usual relation for the continuum limit of a coupling.

However, for our purposes it is simpler to assume that  $\omega$  is a linear function of its index, so that  $\omega = as$  and  $s = i/N$ . The function  $\gamma(s)$  is defined as  $\gamma(s) = \gamma_i$ . More complicated situations can be accommodated by absorbing a density of states factor into  $\gamma$ . In particular, the function  $|\gamma(s)|^2$  would be systematically replaced by  $|\gamma(\omega)|^2 \rho(\omega)$ . With the linearity assumption, equation (2.4) becomes

$$\lambda - h_0 = \int_0^1 \frac{|\gamma(s)|^2}{\lambda - as} ds. \tag{2.8}$$

The function  $\lambda \rightarrow \int_0^1 ds |\gamma(s)|^2 / (\lambda - as)$  is defined for  $\lambda < 0$  and  $\lambda > a$  and decreases in magnitude as  $\lambda$  moves away from that interval. Let us consider the eigenvalues in turn.

2.2.1.a. *The negative root  $\lambda_0$ .* If

$$-h_0 > \int_0^1 \frac{|\gamma(s)|^2}{-as} ds \tag{2.9}$$

then equation (2.8) has a root  $v_0 < 0$ . Obviously this is independent of  $N$ . Returning to the discrete problem, equation (2.4), we see that the root  $\lambda_0 < \omega_1 \equiv 0$  of (2.4) remains a finite positive distance† from  $\omega_1$  when  $N \rightarrow \infty$ . We call the associated eigenstate the plasmon mode (see, e.g. [10]). A sufficient condition for (2.9) to be fulfilled is  $\gamma(0) \neq 0$ . However, it can also be satisfied for  $\gamma(0) = 0$ , but with strong coupling or for  $h_0$  near zero.

On the other hand, if

$$-h_0 < \int_0^1 \frac{|\gamma(s)|^2}{-as} ds \tag{2.10}$$

then there is no root of (2.8) less than 0. Condition (2.10) obviously implies that  $\gamma(0) = 0$ . In this case, if we assume that  $c_1 = 0$ , then figure 3 is replaced by figure 4; the matrix  $H$  becomes

$$H = \begin{pmatrix} h_0 & 0 & c_2^* & \cdot & \cdot & \cdot & c_N^* \\ 0 & \omega_1 & & & & & \\ c_2 & & \omega_2 & & \circ & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & \circ & & & & \\ c_N & & & & & & \omega_N \end{pmatrix}$$

so that  $\omega_1$  is in the spectrum of  $H$  when  $c_1$  is 0.

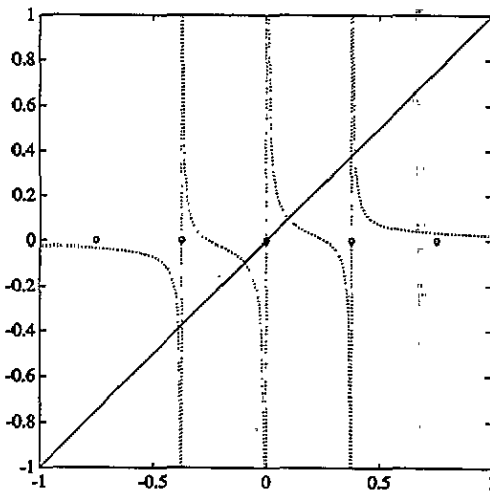


Figure 4. As in figure 3, except that the coefficients,  $c$ , at the edges of the band are zero.

† The term 'finite positive distance' means that the distance is strictly positive for each  $N$  and that its  $N \rightarrow \infty$  limit is also strictly positive.

For  $N$  large enough, we always have the situation shown in figure 4, namely,

$$\lambda_0 = \omega_2 - \epsilon \quad (\omega_2 = a/N).$$

Then equation (2.4) proves that for  $c_1 \equiv 0$ ,

$$\omega_2 - \epsilon - h_0 = \frac{1}{N} \frac{|\gamma_2|^2}{-\epsilon} + \frac{1}{N} \sum_{k=3}^N \frac{|\gamma_k|^2}{-\epsilon - \frac{(k-2)a}{N}}. \tag{2.11}$$

Assume that in the continuum limit

$$|\gamma(s)|^2 \sim |\tilde{\gamma}_0|^2 s^\alpha \quad (\alpha > 0) \tag{2.12}$$

for  $s \rightarrow 0$ . Then we obtain

$$\epsilon \sim \frac{|\tilde{\gamma}_0|^2}{N^{\alpha+1}} \frac{1}{h_0 - \int_0^1 \frac{|\gamma(s)|^2}{as} ds} \tag{2.13}$$

and

$$\lambda_0 \sim \frac{a}{N} - \frac{1}{N^{\alpha+1}} \frac{|\tilde{\gamma}_0|^2}{h_0 - \int_0^1 \frac{|\gamma(s)|^2}{as} ds}. \tag{2.14}$$

2.2.1.b. *The positive root  $\lambda_\infty$ .* The discussion of the largest root is similar. If

$$a - h_0 > \frac{1}{a} \int_0^1 \frac{|\gamma(s)|^2}{1-s} ds \tag{2.15}$$

then there is no root of (2.8) greater than  $a$ . In this case, the condition (2.15) implies  $\gamma(1) = 0$ . Moreover, if  $c_N = 0$ , figure 3 must be modified and  $\omega_{N-1} < \lambda_\infty < \omega_N$ . If we assume

$$|\gamma(s)|^2 \sim |\tilde{\gamma}_1|^2 (1-s)^\beta$$

then we obtain

$$\lambda_\infty = \omega_{N-1} + \frac{|\tilde{\gamma}_1|^2}{N^{\beta+1}} \frac{1}{a - h_0 - \frac{1}{a} \int_0^1 \frac{|\gamma(s)|^2}{1-s} ds}. \tag{2.16}$$

If, on the other hand,

$$a - h_0 < \frac{1}{a} \int_0^1 \frac{|\gamma(s)|^2}{1-s} ds \tag{2.17}$$

then equation (2.8) has a root  $\nu_\infty > a$  and the root  $\lambda_\infty > \omega_N$  of (2.4) stays away from the continuous spectrum  $[0, a]$  of  $H$  when  $N \rightarrow \infty$ .



2.2.2.  $\sigma_\Omega$  is bounded only from below. Now we consider the case where the upper bound  $\omega_N$  lends to infinity while the lower bound  $\omega_1$  stays finite. We still assume  $\omega_1 = 0$ ,  $\omega_k = k\Delta\omega$ , with  $\Delta\omega$  a level spacing that goes to zero. We obtain

$$\lambda - h_0 = \int_0^\infty \frac{|\gamma(s)|^2}{\lambda - s} ds.$$

In the limit  $N \rightarrow \infty$ , the eigenvalue  $\lambda_0 < \omega_1 \equiv 0$  survives and stays away from the continuum  $[0, \infty]$ , provided

$$-h_0 > -\int_0^\infty \frac{|\gamma(s)|^2}{s} ds$$

as in (2.9).

2.2.3. *Subsection summary.* When a level couples to a continuum, despite the fact that the unperturbed energy of that level is within the continuum, there can arise a collective mode of the system with energy *outside* that continuum. The presence of such a level depends on the strength of the coupling (and density of states), its threshold behaviour, and the proximity of the inserted level to the threshold or band edge. For example, the condition for having a level with energy *below* the original continuum is given by (2.9).

2.2.4. *The eigenvalues  $\lambda_j$ ,  $1 \leq j \leq N-1$ .* In this subsection we consider the other eigenvalues of  $H$ . For  $N \rightarrow \infty$ , the roots  $\lambda_1, \dots, \lambda_{N-1}$  of equation (2.4) become the continuous spectrum of  $H$ . Between each pair of levels of  $\Omega$  there is an eigenvalue  $\lambda$ . When we later deal with the time dependence of the decay we will need to know where between these levels the eigenvalue falls.

It is convenient to define

$$F(\lambda) \equiv \lambda - h_0 - \sum_{k=1}^N \frac{|c_k|^2}{\lambda - \omega_k}. \quad (2.18)$$

The eigenvalues we seek are roots,  $F(\lambda) = 0$ . We focus attention on a particular one, say that which lies between  $\omega_j$  and  $\omega_{j+1}$ . This eigenvalue is written as

$$\lambda_j \equiv \frac{\omega_{j+1} + \omega_j}{2} + \theta_j \frac{\omega_{j+1} - \omega_j}{2} \quad (2.19)$$

so that  $-1 < \theta_j < 1$ . Before making the estimate we consider a continuum version of  $F$ , to be called  $G$ . Define

$$G(\lambda) \equiv \lambda - h_0 - P \int_0^a d\omega \frac{f(\omega)}{\lambda - \omega} \quad (2.20)$$

where  $f(\omega) = \rho(\omega)|\gamma(\omega)|^2$  and  $P$  is principal value. Let  $\lambda_0$  be a root of this,  $G(\lambda_0) = 0$ . Unlike  $F(\lambda) = 0$ , this equation has few roots, typically one of them, which we here assume to be the case.

For the function  $F$  we could take a continuum limit, except that this would break down in the neighbourhood of the interval  $[\omega_j, \omega_{j+1}]$ . For the principal value one integrates to within equal distances from the singularity; on the other hand, the way in which the

eigenvalue  $\lambda_j$  manages to be different from  $\lambda_0$  is by having the endpoints of integration displaced from the centre of the interval. We will see this explicitly.

For  $\lambda \in (\omega_j, \omega_{j+1})$ , we take a continuum limit where that can be done, and write  $F$  as

$$F(\lambda) = \lambda - h_0 - \left[ \int_0^{\omega_j} + \int_{\omega_{j+1}}^a \right] \frac{f(\omega)}{\lambda - \omega}. \tag{2.21}$$

We now add and subtract integrals from  $\omega_{j+1}$  to  $\lambda + \delta$  and from  $\lambda - \delta$  to  $\omega_j$ . By letting  $\delta \rightarrow 0$ , a short calculation leads to

$$F(\lambda) = G(\lambda) + f(\lambda) \log \left( \frac{\lambda - \omega_j}{\omega_{j+1} - \lambda} \right) \tag{2.22}$$

for any  $\lambda \in (\omega_j, \omega_{j+1})$ . This immediately yields  $\theta_j = -\tanh[G(\lambda)/2f(\lambda)]$ . For  $\lambda$  near  $\lambda_0$ , we have  $\theta \sim -(\lambda - \lambda_0)G'(\lambda_0)/2f(\lambda_0)$ . In particular, note that at  $\lambda = \lambda_0$ ,  $\theta$  is zero.

### 3. Structure of the eigenvectors

For large fixed  $N$ , we examine the normalized eigenvectors  $\psi^{(j)}$ . In section 3.1 we consider those associated with the extreme eigenvalues. This breaks into several cases, according to whether or not the eigenvalues stay bounded away from the continuum as  $N \rightarrow \infty$ . In section 3.2 we obtain the form of the other eigenvectors.

#### 3.1. Behaviour of $\psi^{(0)}$ or $\psi^{(\infty)}$

There are several cases and several questions to examine within each.

3.1.1.  $\lambda_0$  stays a finite distance from the quasicontinuum  $\omega_1, \dots, \omega_N$ . This is the case if (2.9) holds (taking  $\omega_1 \equiv 0$ ). Then

$$\sum_{k=1}^N \frac{|c_k|^2}{(\lambda_0 - \omega_k)^2} \rightarrow \int_0^1 \frac{|\gamma(s)|^2}{(\lambda_0 - as)^2} ds$$

which is then finite. From equations (2.6) and (2.7) we deduce that

$$x^{(0)} = O(1) \quad y_k^{(0)} = O\left(\frac{\gamma_k}{\sqrt{N}}\right). \tag{3.1}$$

(Recall that  $\gamma_k$  may tend to zero if  $k \rightarrow 0$  or  $k \rightarrow N$ .) This means that even for  $N \rightarrow \infty$ ,  $x^{(0)}$  does not shrink, and  $\psi^{(0)}$  has significant overlap with the state that we are taking as the initial state for decay, namely,  $x = 1, Y = 0$  (in the notation of section 2).

3.1.2.  $\lambda_0$  tends to the continuum spectrum when  $N \rightarrow \infty$ . In this case the situation is slightly more complicated. We use (2.14) and the fact that  $\omega_k = (k - 1)a/N$ ,  $c_1 = 0$

$$\sum_{k=1}^N \frac{|c_k|^2}{(\lambda_0 - \omega_k)^2} = \frac{|c_2|^2}{\epsilon^2} + \sum_{k \geq 3} \frac{|c_k|^2}{(\epsilon - (k - 2)a/N)^2} \tag{3.2}$$

where  $\epsilon$  was calculated earlier (equation (2.13)) and we found that  $|\epsilon| \propto 1/N^{\alpha+1}$ . In equation (3.2), the first term of the right-hand side is

$$\frac{|c_2|^2}{\epsilon^2} = N^{\alpha+1} |\tilde{\gamma}_0|^{-2} \left( h_0 - \int_0^1 \frac{|\gamma(s)|^2}{as} ds \right)^2$$

and the second term is of order  $N^\alpha$ . In particular, from (2.6) we deduce

$$x^{(0)} \sim \frac{|\tilde{\gamma}_0|}{N^{\frac{1+\alpha}{2}} \left( h_0 - \int_0^1 \frac{|\gamma(s)|^2}{as} ds \right)} \tag{3.3}$$

and because

$$y_k^{(0)} = \frac{1}{N} \frac{x^{(0)} \gamma_k}{\lambda_0 - \omega_k}$$

we obtain

$$y_k^{(0)} \sim \frac{1}{N^{(3+\alpha)/2}} \frac{|\tilde{\gamma}_0|}{\left( h_0 - \int_0^1 \frac{|\gamma(s)|^2}{as} ds \right)} \frac{\gamma_k}{\left( -\epsilon - \frac{(k-2)a}{N} \right)} \tag{3.4}$$

Recall here that  $\gamma_1 = 0$  and  $\epsilon$  is given by (2.13). In the same manner we can treat the behaviour of  $\psi^{(\infty)}$  (assuming a finite continuous spectrum  $[0, a]$ ).

We note here a surprising effect. In equation (3.3),  $x^{(0)}$  is of the order of  $O(N^{-\frac{1+\alpha}{2}})$ , while for  $\alpha = 0$  it is of order unity. There is no contradiction, for the following reasons.

(i) When there is a plasmon mode, in particular for  $\alpha = 0$ , there is a different estimate of the eigenvector.

(ii) Here equations (3.3) and (3.4) are derived under the hypothesis that there is no plasmon mode, in particular  $\alpha = 0$ . As we have seen,  $\lambda_0$  is now in the interval  $[\omega_1, \omega_2]$  and the component  $x^{(0)}$  of the corresponding eigenvector should be compared to that of an eigenvector of eigenvalue  $\lambda_k$  in the continuum. We shall see in (3.10) that if  $\gamma_k \neq 0$ ,  $x^{(k)} \sim O(N^{-1/2})$ . So there is no contradiction with (3.3) because when  $\alpha \rightarrow 0$ , equation (3.3) tells us that  $x^{(0)}$  would also be  $O(N^{-1/2})$ .

3.1.3.  $\lambda_\infty$  stays a finite distance from  $a$ . This is the case if (2.17) holds. Then

$$\sum_{k=1}^N \frac{c_k^2}{(\lambda_\infty - \omega_k)^2} \rightarrow \int_0^1 \frac{|\gamma(s)|^2}{(\lambda_\infty - as)^2} ds$$

and we deduce

$$x^{(\infty)} = O(1) \quad y_k^{(\infty)} = O\left(\frac{\gamma_k}{\sqrt{N}}\right) \tag{3.5}$$

3.1.4.  $\lambda_\infty$  tends to the continuous spectrum when  $N \rightarrow \infty$ . This is the case if (2.15) holds. Then equation (3.2) must be modified to

$$\sum_{k=1}^N \frac{|c_k|^2}{(\lambda_\infty - \omega_k)^2} = \frac{|c_{N-1}|^2}{\epsilon^2} + \sum_{1 \leq k < N-1} \frac{|c_k|^2}{\left(\epsilon + \frac{(N-1-k)a}{N}\right)^2} \tag{3.6}$$

where  $\epsilon$  is  $\lambda_\infty - \omega_{N-1}$  and is given by (2.16), namely,

$$\epsilon = \frac{|\tilde{\gamma}_1|^2}{N^{\beta+1}} \frac{1}{a - h_0 - \frac{1}{a} \int_0^1 \frac{|\gamma(s)|^2}{1-s} ds} \tag{3.7}$$

Again in (3.6) the first term on the right-hand side is dominant and we conclude

$$x^{(\infty)} \sim \frac{|\tilde{\gamma}_1|}{N^{(1+\beta)/2} \left(a - h_0 - \frac{1}{a} \int_0^1 \frac{|\gamma(s)|^2}{1-s} ds\right)} \tag{3.8}$$

$$y_k^{(0)} \sim \frac{|\tilde{\gamma}_1|}{N^{(1+\beta)/2} \left(a - h_0 - \frac{1}{a} \int_0^1 \frac{|\gamma(s)|^2}{1-s} ds\right)} \frac{\gamma_k}{\left(\epsilon + \frac{N-k-1}{N}a\right)} \tag{3.9}$$

where  $\epsilon$  is now given by (3.7).

3.2. Behaviour of  $\psi^{(k)}$ ,  $0 < k < \infty$

Consider an eigenvalue  $\lambda_k$  with  $\omega_k < \lambda < \omega_{k+1}$ , and take  $\gamma_k \neq 0$ . Then

$$\frac{1}{N} \sum_{j=1}^N \frac{|\gamma_j|^2}{(\lambda_k - \omega_j)^2} \geq |\gamma_k|^2 N.$$

As a result we can write  $x^{(k)} = O(1/\sqrt{N})$  as

$$x^{(k)} = \frac{\xi^{(k)}}{\sqrt{N}}. \tag{3.10}$$

Near the edges of the continuous spectrum, it may happen that  $\gamma_k$  tends to 0. In this case, the formula (3.10) is still valid but  $\xi^{(k)}$  will tend to infinity. For example, assume, as in (2.12), that

$$|\gamma(s)|^2 \sim |\tilde{\gamma}_0|^2 s^\alpha$$

for  $s \rightarrow 0$ . This means that for fixed  $k$  and  $N \rightarrow \infty$

$$|\gamma_k|^2 \sim |\tilde{\gamma}_0|^2 \left(\frac{k}{N}\right)^\alpha$$

and then

$$\frac{1}{N} \sum_{j=1}^N \frac{|\gamma_j|^2}{(\lambda_k - \omega_j)^2} = \frac{|\tilde{\gamma}_0|^2}{N} \frac{(k/N)^\alpha}{(\lambda_k - \omega_k)^2} + \frac{|\tilde{\gamma}_0|^2}{N} \sum_{\substack{j=1 \\ j \neq k}}^N \frac{(j/N)^\alpha}{(j-k)a/N}.$$

In the last equation, the series on the right-hand side is bounded when  $N \rightarrow \infty$  while the first term is greater than  $O(k^\alpha N^{1-\alpha})$ . In particular, we have for fixed  $k$  and large  $N$

$$|x^{(k)}| = O\left(\left(\frac{N}{k}\right)^{\alpha/2} \frac{1}{\sqrt{N}}\right). \quad (3.11)$$

In the same manner, at the other extremity of the band, namely for  $x^{(N-k)}$ , we have

$$|\gamma(s)|^2 \sim |\tilde{\gamma}_1|^2 (1-s)^\beta$$

and

$$|x^{(N-k)}| = O\left(\left(\frac{N}{N-k}\right)^{\beta/2} \frac{1}{\sqrt{N}}\right) \quad (3.12)$$

again for fixed  $k$  and large  $N$ . This means that the function  $\xi^{(k)}$  defined by (3.10) has the following behaviour near the band edges:

$$\xi^{(k)} = O\left(\left(\frac{N}{k}\right)^{\alpha/2}\right) \quad \xi^{(N-k)} = O\left(\left(\frac{N}{N-k}\right)^{\beta/2}\right) \quad (3.13)$$

for fixed  $k$  and large  $N$ . The components  $y_j^{(k)}$  of  $\psi^{(k)}$  are given by

$$y_j^{(k)} = \frac{\xi^{(k)} \gamma_k}{N^{\beta/2} (\lambda_k - \omega_j)}. \quad (3.14)$$

#### 4. Time dependence: the decay

Using the information we have developed on the spectrum and eigenfunctions we can gain a detailed picture of the decay, in particular, the reasons for the failure of this system to attain full decay in the case where the thresholds are insufficiently smooth, the coupling too strong, or the external level too close to threshold. Our main result concerns the modification of—or defect in—decay when the introduction of the new level creates a plasmon mode.

##### 4.1. Decay of the state $|0\rangle$

We wish to find the time dependence of an initial vector  $|0\rangle$  with components

$$|0\rangle = \begin{pmatrix} 1 \\ Y^{[0]} \end{pmatrix} \quad \text{with} \quad (Y^{[0]})_j = 0 \quad j = 1, \dots, N.$$

It is clear that

$$|0\rangle = x^{(0)} \psi^{(0)} + x^{(\infty)} \psi^{(\infty)} + \sum_{k=1}^{N-1} x^{(k)} \psi^{(k)}$$

and

$$e^{-itH} |0\rangle = x^{(0)} e^{-it\lambda_0} \psi^{(0)} + x^{(\infty)} e^{-it\lambda_\infty} \psi^{(\infty)} + \sum_{k=1}^{N-1} x^{(k)} e^{-it\lambda_k} \psi^{(k)}.$$

The decay of  $|0\rangle$  is obtained by considering

$$\langle 0|e^{-itH}|0\rangle = |x^{(0)}|^2 e^{-it\lambda_0} + |x^{(\infty)}|^2 e^{-it\lambda_\infty} + \sum_{k=1}^{N-1} |x^{(k)}|^2 e^{-it\lambda_k}. \tag{4.1}$$

We begin by studying the behaviour of the last series in the right-hand side of (4.1). We call this sum  $\Phi(t)$  and we examine both its moderate and long-time behaviour.

The moderate time behaviour possesses an independent interest in that we will show how the usual exponential decay *does* emerge. In the usual Wigner–Weisskopf method one uses Laplace transformations, and the assumptions that lead to exponential decay are most conveniently expressed in terms of discarded line integrals and complex singularities that arise in the inverting of the Laplace transform. In our expressions the same general behaviour must emerge from the eigenfunction expansion, but one’s justification resides in the neglect of ostensibly different quantities.

We thus consider intermediate times, that is, well below the Poincaré recurrence times (of the order of the inverse level spacing). Our presentation begins from  $\Phi(t)$ , the series in (4.1) (i.e. the total time dependence, absent the plasmon modes). We make use of (2.6) to obtain

$$\Phi(t) \equiv \sum_{k=1}^{N-1} |x^{(k)}|^2 e^{-it\lambda_k} = \sum_{k=1}^{N-1} \frac{e^{-it\lambda_k}}{\left(1 + \sum_{i=1}^N \frac{|c_i|^2}{(\lambda_i - \omega_i)^2}\right)}. \tag{4.2}$$

Note that this is

$$\sum_{k=1}^{N-1} \frac{e^{-it\lambda_k}}{F'(\lambda_k)}$$

with  $F$  given in (2.18).

In section 2 we further found that for  $\lambda \in [\omega_j, \omega_{j+1}]$ ,  $F$  could be approximated by

$$F(\lambda) = G(\lambda) + f(\lambda) \log\left(\frac{\lambda - \omega_j}{\omega_{j+1} - \lambda}\right)$$

(this was (2.22)) where  $G(\lambda)$  is the continuum approximation for  $F$  given by (2.20) and whose root is  $\lambda_0$ . The derivative of  $F$  is then given by

$$F'(\lambda) = G'(\lambda) + \frac{4f(\lambda)}{(\omega_{j+1} - \omega_j) \cosh^2[G(\lambda)/2f(\lambda)]}$$

The denominator in the second term of  $F'$  is essentially  $d\omega$  or  $d\lambda$  so that this second term will have  $O(N)$  growth when taking the continuum limit. It therefore dominates the  $N$ -independent term  $G'$ . It follows that

$$\Phi(t) = \int d\lambda \frac{e^{-it\lambda}}{4f(\lambda) \cosh^2[G(\lambda)/2f(\lambda)]}$$

Recalling that  $G(\lambda_0) = 0$ , we see that the integrand is peaked at  $\lambda_0$ . If one takes further steps of approximation the ‘Golden Rule’ formula can be derived. Specifically, for  $\lambda$  near  $\lambda_0$  write  $G(\lambda) \approx (\lambda - \lambda_0)G'(\lambda_0)$ . It’s amusing that one *cannot* approximate  $\cosh^2 u$

by  $1 + u^2$ , for otherwise the transition probability misses a factor  $\pi$ . Recalling [12] that  $\int_0^\infty dx \cos ux / \cosh^2 vx = u\pi / [2v^2 \sinh(u\pi/2v)]$ , we see that  $|\Phi|^2$  has a decay rate  $2\pi f(\lambda_0) / G'(\lambda_0)$ . For small coupling  $G'(\lambda_0) \approx 1$ . This gives the 'Golden Rule'. Note too that the difference between  $\lambda_0$  and  $h_0$  is what one generally calls the shift in the real part of the energy.

There is much that is similar in this derivation to the usual Laplace transform methods and yet there is much that is different. What one gets in that situation is an integral of the form  $\int d\lambda \exp(-it\lambda) / \tilde{G}(\lambda)$ , with  $\tilde{G}$  a different continuum limit of  $F$ , namely that in which one takes the principal value as well as the  $i\pi\delta(\lambda - \omega)$  that arises from the singularity. Then to arrive at exponential decay one must justify keeping all but the pole contribution in performing the Laplace inversion. In our expression there is no need to explore the complex singularity structure of the integrand (except possibly for elegant proofs of the formula from [12]). Furthermore, even in the absence of plasmon modes, there is no guarantee of perfect exponential behaviour (in fact, the dominated evolution effect, also known as the quantum Zeno effect, is built into this formula, as is evident from the quoted integral).

We next consider the very long time behaviour. Define

$$\xi(s) = \xi^{(k)} \quad \text{for } s = \frac{k\Delta\omega}{N}.$$

Then for  $N$  large

$$\Phi(t) = \sum_{k=1}^{N-1} |x^{(k)}|^2 e^{-it\lambda_k} \sim \int_0^1 |\xi(s)|^2 e^{-its\Delta\omega} ds.$$

If we assume that  $\gamma(s)$  vanishes at the band edges,  $|\xi(s)|^2$  has a singularity at the band edges given by (3.13):

$$|\xi(s)|^2 \sim \begin{cases} s^\alpha & \text{for } s \text{ near } 0 \\ (1-s)^{-\beta} & \text{for } s \text{ near } 1. \end{cases}$$

This means that for large  $t$ , the behaviour of the integral is

$$\int_0^1 |\xi(s)|^2 e^{itsa} ds \sim K t^{-1+\max(\alpha,\beta)} \quad (4.3)$$

where  $K$  is some constant and where we take the exponent giving the slowest decay due to the contribution of the singularities of  $|\xi(s)|^2$  at the extremities 0 or 1 of the interval.

These power-law decays are similar to results known from other analyses [1-3]. An intuitive physical understanding of this behaviour is given by Newton in terms of the domination of late, distant observations by extremely slow early decay products. This explanation presupposes translational modes, to which our abstract model does not make explicit reference (although the threshold density of states, related to our ' $\alpha$ ' will reflect those modes). In any case, this late stage decay behaviour has not been seen physically. From the estimates of Newton [2] this is not surprising.

If  $\gamma(s)$  does not vanish at the band edge,  $\xi(s)$  has no singularity there and

$$\int_0^1 |\xi(s)|^2 e^{-itsa} ds \sim K t^{-1}.$$

This is a particular case of (4.3) with  $\alpha = \beta = 0$ .

4.2. Summary of decay behaviour

We can now provide a complete discussion of the decay of  $|0\rangle$ .

4.2.1.a. Case 1.  $\lambda_0$  and  $\lambda_\infty$  stay away from the band edges 0 and  $a$  respectively. By equations (3.1) and (3.5),  $x^{(0)}$  and  $x^{(\infty)}$  stay bounded, and we have oscillatory behaviour.

$$\langle 0 | e^{-itH} | 0 \rangle \sim |x^{(0)}|^2 e^{-it\lambda_0} + |x^{(\infty)}|^2 e^{-it\lambda_\infty} + \Phi(t)$$

where  $\Phi(t)$  may be exponential or power law, depending on  $t$ . Notice that these oscillations have frequencies  $\lambda_0$  and  $\lambda_\infty$ , so that their period is much shorter than the Poincaré recurrence time of the system (which is of order  $N$ ).

4.2.1.b. Case 2.  $\lambda_0$  stays away from the band edge 0, but  $\lambda_\infty$  tends to the continuous spectrum. In this case,  $x^{(0)}$  is bounded by (3.1), but by (3.8)  $|x^{(\infty)}|^2$  is of order  $N^{-(1+\beta)}$  and we have oscillatory behaviour

$$\langle 0 | e^{-itH} | 0 \rangle = |x^{(0)}|^2 e^{-it\lambda_0} + O(N^{-(1+\beta)}) + \Phi(t).$$

4.2.1.c. Case 3. Both  $\lambda_0$  and  $\lambda_\infty$  tend to the continuous spectrum. By equations (3.3) and (3.8),  $|x^{(0)}|^2$  and  $|x^{(\infty)}|^2$  tend to zero as powers of  $N$ , and

$$\langle 0 | e^{-itH} | 0 \rangle = O(N^{-(1+\alpha)}) e^{-it\lambda_0} + O(N^{-(1+\beta)}) e^{-it\lambda_\infty}.$$

The oscillatory behaviour is negligible.

We see therefore that when a level persists outside  $\sigma_\Omega$  there is non-exponential decay. This required more than just the existence of a level. An important consideration was that the coefficients of standard initial wavefunctions automatically had significant components, non vanishing in the large- $N$  limit, in the plasmon modes.

4.3. Evolution of a state of the continuous spectrum

Let us now consider the state

$$|j\rangle = \begin{pmatrix} 0 \\ Y^{[j]} \end{pmatrix} \quad \text{with} \quad (Y^{[j]})_k = \delta_{jk} \quad k = 1, \dots, N.$$

We take  $j$  to be within the continuous spectrum. We have

$$|j\rangle = y_j^{(0)*} \psi^{(0)} + y_j^{(\infty)*} \psi^{(\infty)} + \sum_{k=1}^{N-1} y_j^{(k)*} \psi^{(k)}$$

and

$$\langle 0 | e^{-itH} | j \rangle = x^{(0)} y_j^{(0)*} e^{-it\lambda_0} + x^{(\infty)} y_j^{(\infty)*} e^{-it\lambda_\infty} + \sum_{k=1}^{N-1} x^{(k)} y_j^{(k)*} e^{-it\lambda_k}. \tag{4.4}$$

If  $\lambda_0$  stays away from the continuous spectrum, by equation (3.1), we deduce

$$x^{(0)} y_j^{(0)*} = O(1/\sqrt{N}).$$



If  $\lambda_0$  tends to the continuous spectrum, by (3.3) and (3.4) we deduce

$$x^{(0)} y_j^{(0)*} = O(1/N^{2+\alpha}).$$

The same discussion holds for  $x^{(\infty)} y_j^{(\infty)*}$ . Moreover, the last sum in (4.4) is estimated as

$$\sum_{k=1}^{N-1} x^{(k)} y_j^{(k)*} e^{-ir\lambda_k} = \frac{1}{N} \sum_{k=1}^{N-1} |\xi^{(k)}|^2 \frac{\gamma_k^*}{N(\lambda_k - \omega_j)} e^{-ir\lambda_k}.$$

The dominant behaviour is due to the terms near  $j$  (at least if  $j$  is well inside the continuum) because at the band edge  $|\xi^{(k)}|^2 \gamma_k^*$  has a singularity  $s^{-\alpha/2}$  or  $(1-s)^{-\alpha/2}$ . It is easy to deduce

$$\left| \sum_{k=1}^{N-1} x^{(k)} y_j^{(k)*} e^{-ir\lambda_k} \right| = O\left(\frac{\log N}{N}\right).$$

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